On a Koolen – Park inequality and Terwilliger graphs

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Abstract

J.H. Koolen and J. Park have proved a lower bound for intersection number c_2 of a distance-regular graph Γ. Moreover, they showed that the graph Γ which attains the equality in this bound is a Terwilliger graph. We prove that Γ is the icosahedron, the Doro graph or the Conway-Smith graph, if equality is attained and $c_2 \geq 2$.

Key Words: Terwilliger graphs, distance-regular graphs

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1 Introduction

Let Γ be a distance-regular graph with degree k and diameter at least 2. Let c be maximal such that for each vertex $x \in \Gamma$ and every pair of nonadjacent vertices y, z of $\Gamma_1(x)$, there exists a c-coclique in $\Gamma_1(x)$ containing y, z. In [1], J.H. Koolen and J. Park have shown that the following bound holds:

$$c_2 - 1 \geq \max\{\frac{c'(a_1+1)-k}{\binom{c'}{2}} \mid 2 \leq c' \leq c\}, \tag{1}$$

and equality implies Γ is a Terwilliger graph. (For definitions see sections 2 and 3.)

The similar inequality for a distance-regular graph with c-claw was proved by C.D. Godsil, see [2]. J.H. Koolen and J. Park [1] have noted that the bound (1) is met exactly for all known examples of Terwilliger graphs. We recall that only three examples of distance-regular Terwilliger graphs with $c_2 \geq 2$ are known: the icosahedron, the Doro graph and the Conway-Smith graph.

In this paper, we will show that the distance-regular graph Γ with $c_2 \geq 2$ which attains the equality in (1) is a known Terwilliger graph.

2 Definitions and preliminaries

We consider only finite, undirected graphs without loops or multiple edges. Let Γ be a connected graph. The *distance* $\mathrm{d}(u, w)$ between any two vertices u and w of Γ is the length of a shortest path from u to w in Γ. The *diameter* $\mathrm{diam}(\Gamma)$ of Γ is the maximal distance occuring in Γ.

For a subset A of the vertex set of Γ, we will also write A for the subgraph of Γ induced by A. For a vertex u of Γ, define $\Gamma_i(u)$ to be the set of vertices which are at distance i from u ($0 \leq i \leq \mathrm{diam}(\Gamma)$). The subgraph $\Gamma_1(u)$ is called the *local graph* of a vertex u and the *degree* of u is the number of neighbours of u, i.e. $|\Gamma_1(u)|$.

For two vertices $u, w \in \Gamma$ with $\mathrm{d}(u, w) = 2$, the subgraph $\Gamma_1(u) \cap \Gamma_1(w)$ is called *μ-subgraph* of vertices u, w. We say the number $\mu(\Gamma)$ is *well-defined*, if each μ-subgraph occuring in Γ contains the same number of vertices which is equal to $\mu(\Gamma)$.

Let Δ be a graph. A graph Γ is *locally* Δ, if, for all $u \in \Gamma$, the subgraph $\Gamma_1(u)$ is isomorphic to Δ. A graph is *regular* with degree k, if the degree of each its vertex is k.

A connected graph Γ with diameter $d = \mathrm{diam}(\Gamma)$ is *distance-regular*, if there are integers b_i, c_i $(0 \leq i \leq d)$ such that for any two vertices $u, w \in \Gamma$ with $\mathrm{d}(u, w) = i$, there are exactly c_i neighbours of w in $\Gamma_{i-1}(u)$ and b_i neighbours of w in $\Gamma_{i+1}(u)$ (we assume that $\Gamma_{-1}(u)$ and $\Gamma_{d+1}(u)$ are empty sets). In particular, distance-regular graph Γ is regular with degree b_0, $c_1 = 1$ and $c_2 = \mu(\Gamma)$. For each vertex $u \in \Gamma$ and $0 \leq i \leq d$, the subgraph $\Gamma_i(u)$ is regular with degree $a_i = b_0 - b_i - c_i$. The numbers b_i, c_i $(0 \leq i \leq d)$ are called the *intersection numbers* and the array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, is called the *intersection array* of distance-regular graph Γ.

A graph Γ is *amply regular* with parameters (v, k, λ, μ), if Γ has v vertices, it is regular with degree k and the following two conditions hold:

i) for each pair of adjacent vertices $u, w \in \Gamma$, the subgraph $\Gamma_1(u) \cap \Gamma_1(w)$ contains exactly λ vertices;

ii) $\mu = \mu(\Gamma)$ is well-defined.

An amply regular graph with diameter 2 is called a *strongly regular* graph and it is a distance-regular graph. A distance-regular graph is an amply regular graph with parameters $k = b_0$, $\lambda = b_0 - b_1 - 1$ and $\mu = c_2$.

Recall that a $(c-)$clique (or *complete* graph) is a graph (on c vertices) in which every pair of its vertices is adjacent. A $(c-)$coclique is a graph (on c vertices) in which every pair of its vertices is not adjacent.

Let Γ be a strongly regular graph with parameters $(v, k, \lambda, 1)$. There are integers r and s such that the local graph of each vertex of Γ is the disjoint union of r copies of s-clique. Futhermore, $v = 1 + rs + s^2 r(r-1)$, $k = rs$ and $\lambda = s - 1$. We denote the set of strongly regular graphs with such parameters by $\mathcal{F}(s, r)$.

Any graph of $\mathcal{F}(1, r)$, i.e. a strongly regular graph with $\lambda = 0$ and $\mu = 1$, is called a *Moore* strongly regular graph. It is well known (see Chapter 1 [3]) that any Moore strongly regular graph has degree 2, 3, 7 or 57. The graphs with degree 2, 3 and 7 are the pentagon, the Petersen graph and the Hoffman - Singleton graph, respectively. Whether a Moore graph with degree 57 exists is an open problem.

Lemma 2.1 *Suppose that $\mathcal{F}(s, r)$ is nonempty set of graphs. Then $s+1 \leq r$.*

Proof. Let Γ be a graph of $\mathcal{F}(s, r)$. We may choose vertices u and w of Γ with $\mathrm{d}(u, w) = 2$. Let x be a vertex of $\Gamma_1(u) \cap \Gamma_1(w)$. Then the subgraph $\Gamma_1(w) - (\Gamma_1(x) \cup \{x\})$ contains a coclique of size at most $r-1$. Let us consider a

s-clique of $\Gamma_1(u) - \Gamma_1(w)$ on vertices $y_1, y_2, .., y_s$. The subgraph $\Gamma_1(w) \cap \Gamma_1(y_i)$ $(1 \leq i \leq s)$ contains a single vertex z_i. The vertices $z_1, z_2, .., z_s$ are mutually nonadjacent and distinct. Hence, $s \leq r - 1$. The lemma is proved. ∎

3 Terwilliger graphs

In this section we give a definition of Terwilliger graphs and some useful facts concerning them.

A *Terwilliger graph* is a connected noncomplete graph Γ such that $\mu(\Gamma)$ is well-defined and each μ-subgraph occuring in Γ is a complete graph (hence, there are no induced quadrangles in Γ). If $\mu(\Gamma) > 1$, then, for each vertex $u \in \Gamma$, the local graph of u will also be a Terwilliger graph with diameter 2 and $\mu(\Gamma_1(u)) = \mu(\Gamma) - 1$.

For an integer $\alpha \geq 1$, a α-clique extension of a graph $\bar{\Gamma}$ is the graph Γ obtained from $\bar{\Gamma}$ by replacing each vertex $\bar{u} \in \bar{\Gamma}$ by a clique U of α vertices, where for any $\bar{u}, \bar{w} \in \bar{\Gamma}$, $u \in U$ and $w \in W$, $\bar{u}$ and $\bar{w}$ are adjacent if and only if u and w are adjacent.

Lemma 3.1 *Let Γ be an amply regular Terwilliger graph with parameters (v, k, λ, μ), where $\mu > 1$. There is the number α such that the local graph of each its vertex is the α-clique extension of a strongly regular Terwilliger graph with parameters $(\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})$, where*

$$\bar{v} = k/\alpha, \ \bar{k} = (\lambda - \alpha + 1)/\alpha, \ \bar{\mu} = (\mu - 1)/\alpha,$$

and $\alpha \leq \bar{\lambda} + 1$. In particular, if $\bar{\lambda} = 0$, then $\alpha = 1$.

Proof. The result follows from [3, Theorem 1.16.3]. ∎

We know only three examples of amply regular Terwilliger graphs with $\mu \geq 2$. All of them are unique distance-regular locally Moore graphs:

(1) the icosahedron with intersection array $\{5, 2, 1; 1, 2, 5\}$ is locally pentagon.

(2) the Doro graph with intersection array $\{10, 6, 4; 1, 2, 5\}$ and the Conway-Smith graph with intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$ are locally Petersen graphs.

In [4], A. Gavrilyuk and A. Makhnev have shown that a distance-regular locally Hoffman – Singleton graph has intersection array $\{50, 42, 9; 1, 2, 42\}$

or $\{50, 42, 1; 1, 2, 50\}$ and hence it is a Terwilliger graph. Whether the graphs with these intersection arrays exist is an open question.

Lemma 3.2 *Let Γ be a Terwilliger graph. Suppose that, for an integer $\alpha \geq 1$, the local graph of each its vertex is the α-clique extension of a Moore strongly regular graph Δ. Then $\alpha = 1$ and one of the following holds:*

(1) Δ *is the pentagon and* Γ *is the icosahedron*;

(2) Δ *is the Petersen graph and* Γ *is the Doro graph or the Conway-Smith graph*;

(3) Δ *is the Hoffman – Singleton graph or a graph with degree* 57*, in both cases diameter of* Γ *is at least* 3*.*

Proof. It is easy to see that the graph Γ is amply regular. By Lemma 3.1, we have $\alpha = 1$. The statements (1) and (2) follow from [3, Proposition 1.1.4] and [3, Theorem 1.16.5], respectively.

If the graph Δ is the Hoffman – Singleton graph and diameter of Γ is 2, then Γ is strongly regular with parameters (v, k, λ, μ), where $k = 50$, $\lambda = 7$ and $\mu = 2$. By [3, Theorem 1.3.1], the eigenvalues of Γ are k and the roots of the quadratic equation $x^2 + (\mu - \lambda)x + (\mu - k) = 0$. The roots of the equation $x^2 - 5x - 48 = 0$ are not integers, that is impossible. In the remained case, when Δ is regular with degree 57, we will get the same contradiction. The lemma is proved. ∎

The next lemma is useful in the proof of Theorem 4.2 (see Section 4).

Lemma 3.3 *Let Γ be a strongly regular Terwilliger graph with parameters (v, k, λ, μ). Suppose that, for an integer $\alpha \geq 1$, the local graph of each its vertex is the α-clique extension of a strongly regular graph Δ with parameters $(\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})$. Then $\bar{k} - \bar{\lambda} - \bar{\mu} > 1$ implies that $k - \lambda - \mu > 1$.*

Proof. We have $k = \alpha(1 + \bar{k} + \bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu})$, $\lambda = \alpha\bar{k} + \alpha - 1$ and $\mu = \alpha\bar{\mu} + 1$. If $\bar{k} - \bar{\lambda} - \bar{\mu} > 1$, then $\bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu} > \bar{k}$ and this implies that $k - \lambda - \mu = \alpha(\bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu} - \bar{\mu}) > \alpha(\bar{k} - \bar{\mu}) > \alpha(\bar{\lambda} + 1) \geq 1$. ∎

4 Koolen – Park inequality

In this section, we consider the bound (1) and classify distance-regular graphs with $c_2 \geq 2$ which attain this bound.

The next proposition is a slight generalization of [1, Proposition 3]. J.H. Koolen and J. Park [1, Proposition 3] formulated the next proposition for distance-regular graphs. We generalize it to amply regular graphs. (Our proof is similar to the one in J.H. Koolen and J. Park [1], but we give it for convenience of the reader.)

Proposition 4.1 *Let Γ be an amply regular graph with parameters (v, k, λ, μ) and $c \geq 2$ be maximal such that for each vertex $x \in \Gamma$ and every pair of non-adjacent vertices y, z of $\Gamma_1(x)$, there exists a c-coclique in $\Gamma_1(x)$ containing y, z. Then*

$$\mu - 1 \geq \max\{\frac{c'(\lambda+1) - k}{\binom{c'}{2}} \mid 2 \leq c' \leq c\},$$

and if equality is attained, then Γ is a Terwilliger graph.

Proof. Let $\Gamma_1(x)$ contain a coclique C' on vertices $y_1, y_2, \ldots, y_{c'}$, $c' \geq 2$. Since $\mathrm{d}(y_i, y_j) = 2$, $|\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)| \leq \mu - 1$ holds for all $i \neq j$. Then by the principle of inclusion and exclusion,

$$k = |\Gamma_1(x)| \geq |\cup_{i=1}^{c'} (\Gamma_1(x) \cap (\Gamma_1(y_i) \cup \{y_i\}))|$$

$$\geq \sum_{i=1}^{c'} |\Gamma_1(x) \cap (\Gamma_1(y_i) \cup \{y_i\})| - \sum_{1 \leq i < j \leq c'} |\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)|$$

$$\geq c'(\lambda+1) - \binom{c'}{2}(\mu - 1).$$

So,

$$\mu - 1 \geq \frac{c'(\lambda+1) - k}{\binom{c'}{2}}. \tag{2}$$

Note that equality in (2) implies that $\Gamma_1(x) \subseteq \cup_{i=1}^{c'}(\Gamma_1(y_i) \cup \{y_i\})$ holds and we have $|\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)| = \mu - 1$ for all $i \neq j$.

Let c be maximal satisfying the condition of the Proposition 4.1. Then

$$\mu - 1 \geq \max\{\frac{c'(\lambda+1) - k}{\binom{c'}{2}} \mid 2 \leq c' \leq c\}. \tag{3}$$

We may assume that for an integer c'', where $2 \leq c'' \leq c$, equality holds in (3), i.e.

$$\mu - 1 = \frac{c''(\lambda+1) - k}{\binom{c''}{2}} = \max\{\frac{c'(\lambda+1) - k}{\binom{c'}{2}} \mid 2 \leq c' \leq c\}. \tag{4}$$

We will show $c = c''$. For a vertex $x \in \Gamma$ and nonadjacent vertices $y, z \in \Gamma_1(x)$, there exists a c-coclique C in $\Gamma_1(x)$ containing y, z. The equality (4) implies that, for any subset of vertices $\{y_1, y_2, \ldots, y_{c''}\} \subseteq C$, $\Gamma_1(x) \subseteq \cup_{i=1}^{c''}(\Gamma_1(y_i) \cup \{y_i\})$ holds. But if $c'' < c$, then $C \not\subseteq \cup_{i=1}^{c''}(\Gamma_1(y_i) \cup \{y_i\})$, which is the contradiction.

Hence, $c = c''$ and we have $|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(z)| = \mu - 1$ for every pair of nonadjacent vertices $y, z \in \Gamma_1(x)$ and for all $x \in \Gamma$. This implies that each μ-subgraph occuring in Γ is a clique of size μ and Γ is a Terwilliger graph. ∎

We call the inequality (3) *μ-bound*.

Let Γ be an amply regular Terwilliger graph with parameters (v, k, λ, μ). If $\mu = 1$, then the local graph of each its vertex is the disjoint union of $k/(\lambda + 1)$ copies of $(\lambda + 1)$-clique, so equality in μ-bound is attained. If $\mu \geq 2$, then we know only three examples of Γ (see Section 3) with $\mu = 2$ and each of them attains equality in μ-bound:

(1) Γ is the icosahedron. The pentagon contains a 2-coclique and is regular with degree 2, i.e. $c = 2$ and $\lambda = 2$, hence we have $(2\cdot(2+1)-5)/\binom{2}{2} = 1 = \mu - 1$.

(2) Γ is the Doro graph or the Conway-Smith graph. The Petersen graph contains a 4-coclique and is regular with degree 3, hence we have $(4 \cdot (3 + 1) - 10)/\binom{4}{2} = (16 - 10)/6 = 1 = \mu - 1$.

Recall that the Hoffman – Singleton graph contains a 15-coclique. If Γ is an amply regular locally Hoffman – Singleton graph and is a Terwilliger graph, then $\mu = 2$, but equality in μ-bound is not attained.

Theorem 4.2 *Let Γ be an amply regular graph with parameters (v, k, λ, μ) and $\mu > 1$. If Γ attains equality in μ-bound, then $\mu = 2$ and Γ is the icosahedron, the Doro graph or the Conway-Smith graph.*

Proof. By Proposition 4.1, the graph Γ is a Terwilliger graph and, be Lemma 3.1, there is an integer $\alpha \geq 1$ such that the local graph of each its vertex is the α-clique extension of a strongly regular Terwilliger graph with parameters $(\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})$. By Lemma 3.1, we have $k = \alpha\bar{v}$, $\lambda = \alpha\bar{k}+(\alpha-1)$ and $\mu = \alpha\bar{\mu}+1$.

By the assumption on Γ, for a vertex $u \in \Gamma$, the local graph of u contains a c-coclique which attains equality in μ-bound, i.e.

$$\mu(\Gamma) - 1 = \alpha\bar{\mu} = \frac{c(\alpha\bar{k} + (\alpha - 1) + 1) - \alpha\bar{v}}{\binom{c}{2}} = \alpha\frac{c(\bar{k} + 1) - \bar{v}}{\binom{c}{2}}$$

and

$$\bar{\mu} = \frac{c(\bar{k}+1) - \bar{v}}{\binom{c}{2}}.$$

Straightforward,

$$c^2\bar{\mu} - c(\bar{\mu} + 2(\bar{k}+1)) + 2\bar{v} = 0,$$

so

$$c = \frac{(\bar{\mu} + 2(\bar{k}+1)) \pm \sqrt{(\bar{\mu} + 2(\bar{k}+1))^2 - 8\bar{v}\bar{\mu}}}{2\bar{\mu}},$$

and

$$(\bar{\mu} + 2(\bar{k}+1))^2 \geq 8\bar{v}\bar{\mu}.$$

Let the subgraph $\Gamma_1(u)$ be isomorphic to the α-clique extension of a strongly regular Terwilliger graph Δ. The cardinality of the vertex set of Δ is equal to $\bar{v} = 1 + \bar{k} + \bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu}$, hence:

$$(\bar{\mu} + 2(\bar{k}+1))^2 \geq 8(\bar{\mu} + \bar{k}\bar{\mu} + \bar{k}(\bar{k} - \bar{\lambda} - 1)),$$

$$\bar{\mu}^2 + 4 \geq 4\bar{\mu} + 4\bar{k}\bar{\mu} + 4\bar{k}^2 - 8\bar{k}\bar{\lambda} - 16\bar{k}.$$

Next,

$$(\bar{\mu}/2)^2 + 1 \geq \bar{\mu} + \bar{k}\bar{\mu} + \bar{k}^2 - 2\bar{k}\bar{\lambda} - 4\bar{k},$$

$$((\bar{\mu}/2) - (\bar{k}+1))^2 \geq 2\bar{k}(\bar{k} - \bar{\lambda} - 1). \qquad (5)$$

At first, we may assume $\bar{\mu} = 1$. There are integers s, r such that $\Delta \in \mathcal{F}(s,r)$ and $\bar{k} = rs$, $\bar{\lambda} = s - 1$. If $\bar{k} - \bar{\lambda} - 1 \geq \bar{k}/2 + 1$, then $2\bar{k}(\bar{k} - \bar{\lambda} - 1) \geq 2\bar{k}(\bar{k}/2 + 1) = \bar{k}^2 + 2\bar{k}$. It follows from (5) that $(\bar{k} + 1/2)^2 \geq \bar{k}^2 + 2\bar{k}$ and hence $1/4 \geq \bar{k}$, that is impossible. Therefore, $\bar{k} - \bar{\lambda} - 1 < \bar{k}/2 + 1$, i.e. $\bar{k} < 2(\bar{\lambda} + 2)$ holds. Substituting the expressions for $\bar{k}$ and $\bar{\lambda}$ into the previous inequality yields $rs < 2(s+1)$. By Lemma 2.1, we have $s + 1 \leq r$. Hence, $s + 1 \leq r < 2(s+1)/s$ and this implies that $s = 1$, $r \in \{2, 3\}$ and Δ is the pentagon or the Petersen graph. In both cases Theorem 4.2 follows from Lemma 3.2.

Now we may assume $\bar{\mu} > 1$. Since $\bar{\mu} < \bar{k}$, the left side of (5) is at most $\bar{k}^2$. On the other hand, if $\bar{k} - \bar{\lambda} - 1 > \bar{k}/2$ holds, then the right side of (5) is more than $2\bar{k}\bar{k}/2 = \bar{k}^2$, that is impossible. Hence, we have $\bar{k} - \bar{\lambda} - 1 \leq \bar{k}/2$, i.e. $\bar{k} \leq 2(\bar{\lambda} + 1)$.

Since $\bar{\mu} > 1$, there is an integer $\alpha_1 \geq 1$ such that, for a vertex $w \in \Delta$, the subgraph $\Delta_1(w)$ is the α_1-clique extension of a strongly regular Terwilliger graph Σ with parameters $(v_1, k_1, \lambda_1, \mu_1)$, where $v_1 = \dfrac{\bar{k}}{\alpha_1}$, $k_1 = \dfrac{\bar{\lambda} - (\alpha_1 - 1)}{\alpha_1}$, $\mu_1 = \dfrac{\bar{\mu} - 1}{\alpha_1}$. Then the inequality $\bar{k} \leq 2(\bar{\lambda} + 1)$ is equivalent to the inequality $v_1 \leq 2(k_1 + 1)$ and the cardinality of the vertex set of Σ is equal to

$$v_1 = 1 + k_1 + k_1 \frac{(k_1 - \lambda_1 - 1)}{\mu_1}.$$

Next, $v_1 \leq 2(k_1 + 1)$ implies that

$$\frac{k_1(k_1 - \lambda_1 - 1)}{\mu_1} \leq k_1 + 1,$$

so

$$k_1 - \lambda_1 - 1 \leq \mu_1(1 + 1/k_1) < \mu_1 + 1,$$

and

$$k_1 < \lambda_1 + \mu_1 + 2. \tag{6}$$

If $\mu_1 = 1$, then, for certain s_1, r_1, we have $k_1 = r_1 s_1$ and $\lambda_1 = s_1 - 1$. It follows from (6) that $r_1 s_1 < s_1 - 1 + 1 + 2 = s_1 + 2$, $r_1 < 1 + 2/s_1$ and $s_1 = 1$, $r_1 = 2$. Hence, the graph $\Delta_1(w)$ is the α_1-clique extension of the pentagon. By Lemma 3.2, the graph Δ is the icosahedron and diameter of $\Gamma_1(u)$ is 3, that is impossible because Γ is a Terwilliger graph.

Hence, $\mu_1 > 1$. Let us consider a sequence of strongly regular graphs $\Sigma_1 = \Sigma$, $\Sigma_2, \ldots, \Sigma_h$, $h \geq 2$ such that, for an integer $\alpha_{i+1} \geq 1$, the local graph of a vertex in Σ_i is the α_{i+1}-clique extension of a strongly regular Terwilliger graph Σ_{i+1} with parameters $(v_{i+1}, k_{i+1}, \lambda_{i+1}, \mu_{i+1})$, $1 \leq i < h$ and $\mu(\Sigma_h) = 1$, i.e. $\Sigma_h \in \mathcal{F}(s_h, r_h)$ for certain s_h, r_h. The sequence exists by Lemma 3.1.

Assuming that $s_h > 1$, we may note $k_h - \lambda_h - \mu_h = r_h s_h - (s_h - 1) - 1 = s_h(r_h - 1) > 1$. According to Lemma 3.3, we have $k_i - \lambda_i - \mu_i > 1$ for all $1 \leq i \leq h - 1$, which is the contradiction with (6). Hence, $s_h = 1$ and Σ_h is a Moore strongly regular graph. By Lemma 3.2, diameter of Σ_{h-1} is at least 3, which is the contradiction that completes the proof. ∎

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